

Form 5 Chapter 3
Integration
Fully-Worked Solutions

UPSKILL 3.2a

1 (a) $\int -10 \, dx$
 $= -10x + c$

(b) $\int 0 \, dx$
 $= 0x + c$
 $= c$

(c) $\int dx$
 $= 1x + c$
 $= x + c$

(d) $\int 6x \, dx$
 $= \frac{6x^2}{2} + c$
 $= 3x^2 + c$

(e) $\int 6x^2 \, dx$
 $= 6 \left(\frac{x^3}{3} \right) + c$
 $= 2x^3 + c$

(f) $\int \frac{5}{3} x^4 \, dx$
 $= \frac{5}{3} \left(\frac{x^5}{5} \right) + c$
 $= \frac{x^5}{3} + c$

(g) $\int \frac{6}{x^4} \, dx$
 $= \int 6x^{-4} \, dx$
 $= \frac{6x^{-3}}{-3} + c$
 $= -2x^{-3} + c$
 $= -\frac{2}{x^3} + c$

(h) $\int -\frac{2}{5x^3} \, dx$
 $= \int -\frac{2}{5} x^{-3} \, dx$
 $= -\frac{2}{5} \left(\frac{x^{-2}}{-2} \right) + c$
 $= \frac{1}{5x^2} + c$

(i) $\int \frac{3}{x^7} \, dx$
 $= \int 3x^{-7} \, dx$
 $= 3 \left(\frac{x^{-6}}{-6} \right) + c$
 $= -\frac{1}{2x^6} + c$

(j) $\int \frac{5}{3x^6} \, dx$
 $= \int \frac{5}{3} x^{-6} \, dx$
 $= \frac{5}{3} \frac{x^{-5}}{-5} + c$
 $= -\frac{1}{3x^5} + c$

2 (a) $\int (5x^4 + 3x^2 - 4) \, dx$
 $= \frac{5x^5}{5} + \frac{3x^3}{3} - 4x + c$
 $= x^5 + x^3 - 4x + c$

(b) $\int \left(\frac{2}{5} x^3 + \frac{1}{4x^3} \right) \, dx$
 $= \int \left(\frac{2}{5} x^3 + \frac{1}{4} x^{-3} \right) \, dx$
 $= \frac{2x^4}{20} + \frac{1}{4} \left(\frac{x^{-2}}{-2} \right) + c$
 $= \frac{x^4}{10} - \frac{1}{8x^2} + c$

$$\begin{aligned}
 \text{(c)} \quad & \int \left(3 - \frac{4}{x^2} + \frac{6}{x^3} \right) dx \\
 &= \int (3 - 4x^{-2} + 6x^{-3}) dx \\
 &= 3x - 4 \frac{x^{-1}}{-1} + 6 \left(\frac{x^{-2}}{-2} \right) + c \\
 &= 3x + \frac{4}{x} - \frac{3}{x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int \left(2 + \frac{4}{x^2} - \frac{3}{x^4} \right) dx \\
 &= \int (2 + 4x^{-2} - 3x^{-4}) dx \\
 &= 2x + \frac{4x^{-1}}{-1} - \frac{3x^{-3}}{-3} + c \\
 &= 2x - \frac{4}{x} + \frac{1}{x^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{3 (a)} \quad & \int 2x^2(4x-6) dx \\
 &= \int (8x^3 - 12x^2) dx \\
 &= \frac{8x^4}{4} - \frac{12x^3}{3} + c \\
 &= 2x^4 - 4x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int (3x+2)(2x-1) dx \\
 &= \int (6x^2 + x - 2) dx \\
 &= \frac{6x^3}{3} + \frac{x^2}{2} - 2x + c \\
 &= 2x^3 + \frac{x^2}{2} - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int (2-3x)^2 dx \\
 &= \int (4-12x+9x^2) dx \\
 &= 4x - \frac{12x^2}{2} + \frac{9x^3}{3} + c \\
 &= 4x - 6x^2 + 3x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int \left(1 + \frac{2}{x} \right) \left(1 - \frac{2}{x} \right) dx \\
 &= \int \left(1 - \frac{4}{x^2} \right) dx \\
 &= \int (1 - 4x^{-2}) dx \\
 &= x - \frac{4x^{-1}}{-1} + c \\
 &= x + \frac{4}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{4 (a)} \quad & \int \left(\frac{x^4+2}{x^2} \right)^2 dx \\
 &= \int \left(\frac{x^8+4x^4+4}{x^4} \right) dx \\
 &= \int (x^4 + 4 + 4x^{-4}) dx \\
 &= \frac{x^5}{5} + 4x + \frac{4x^{-3}}{-3} + c \\
 &= \frac{x^5}{5} + 4x - \frac{4}{3x^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int \frac{(x+4)(x-4)}{x^2} dx \\
 &= \int \left(\frac{x^2-16}{x^2} \right) dx \\
 &= \int (1-16x^{-2}) dx \\
 &= x - \frac{16x^{-1}}{-1} + c \\
 &= x + \frac{16}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int \frac{3x^3-3x^2+4}{x^2} dx \\
 &= \int (3x-3+4x^{-2}) dx \\
 &= \frac{3x^2}{2} - 3x + \frac{4x^{-1}}{-1} + c \\
 &= \frac{3x^2}{2} - 3x - \frac{4}{x} + c
 \end{aligned}$$

UPSKILL 3.2b

$$\begin{aligned}
 1 \text{ (a)} \int (2x+1)^4 dx &= \frac{(2x+1)^5}{5(2)} + c \\
 &= \frac{(2x+1)^5}{10} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \int (2-x)^3 dx &= \frac{(2-x)^4}{4(-1)} + c \\
 &= -\frac{(2-x)^4}{4} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \int 4(3x-2)^5 dx &= \frac{4(3x-2)^6}{6(3)} + c \\
 &= \frac{2}{9}(3x-2)^6 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \int 3\left(\frac{1}{2}x-4\right)^3 dx &= 3 \frac{\left(\frac{1}{2}x-4\right)^4}{4\left(\frac{1}{2}\right)} + c \\
 &= \frac{3}{2}\left(\frac{1}{2}x-4\right)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \int 2(3x-2)^{-2} dx &= \frac{2(3x-2)^{-1}}{-1(3)} + c \\
 &= -\frac{2}{3}(3x-2)^{-1} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \int 2(4-3x)^{-6} dx &= \frac{2(4-3x)^{-5}}{-5(-3)} + c \\
 &= \frac{2}{15}(4-3x)^{-5} + c
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ (a)} \int \frac{1}{(3x-4)^2} dx &= \int (3x-4)^{-2} dx \\
 &= \frac{(3x-4)^{-1}}{-1(3)} + c \\
 &= -\frac{1}{3(3x-4)} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \int \frac{-2}{(2x-3)^4} dx &= \int -2(2x-3)^{-4} dx \\
 &= \frac{-2(2x-3)^{-3}}{-3(2)} + c \\
 &= \frac{1}{3(2x-3)^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \int \frac{3}{(4x-3)^4} dx &= \int 3(4x-3)^{-4} dx \\
 &= \frac{3(4x-3)^{-3}}{-3(4)} + c \\
 &= -\frac{1}{4(4x-3)^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \int \frac{5}{(4-3x)^3} dx &= \int 5(4-3x)^{-3} dx \\
 &= \frac{5(4-3x)^{-2}}{-2(-3)} + c \\
 &= \frac{5}{6(4-3x)^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \int \frac{3}{4(6-x)^5} dx &= \int \frac{3(6-x)^{-5}}{4} dx \\
 &= \frac{3(6-x)^{-4}}{4(-4)(-1)} + c \\
 &= \frac{3}{16(6-x)^4} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \int \frac{2}{3(4-2x)^2} dx \\
 &= \int \frac{2(4-2x)^{-2}}{3} dx \\
 &= \frac{2(4-2x)^{-1}}{3(-1)(-2)} \\
 &= \frac{1}{3(4-2x)} + c
 \end{aligned}$$

UPSKILL 3.2c

1 $\frac{dy}{dx} = x^2(4x-3)$

$$\begin{aligned}
 y &= \int (4x^3 - 3x^2) dx \\
 &= \frac{4x^4}{4} - \frac{3x^3}{3} + c \\
 &= x^4 - x^3 + c
 \end{aligned}$$

Since the curve passes through the point (2, 5), thus $x = 2$ and $y = 5$.

$$\begin{aligned}
 5 &= 2^4 - 2^3 + c \\
 c &= -3
 \end{aligned}$$

Hence, the equation of the curve is

$$y = x^4 - x^3 - 3.$$

2 $\frac{dy}{dx} = (2x-4) dx$

$$\begin{aligned}
 y &= \int (2x-4) dx \\
 &= \frac{2x^2}{2} - 4x + c \\
 &= x^2 - 4x + c
 \end{aligned}$$

Since the curve passes through the point (3, 0), thus $x = 3$ and $y = 0$.

$$\begin{aligned}
 0 &= 3^2 - 4(3) + c \\
 c &= 3
 \end{aligned}$$

Hence, the equation of the curve is

$$y = x^2 - 4x + 3.$$

3 $\frac{dy}{dx} = \frac{2}{x^2} + 1$

$$\begin{aligned}
 y &= \int \left(\frac{2}{x^2} + 1 \right) dx \\
 &= \int (2x^{-2} + 1) dx \\
 &= \frac{2x^{-1}}{-1} + x + c \\
 &= -\frac{2}{x} + x + c
 \end{aligned}$$

Since the curve passes through the point (1, 8), thus $x = 1$ and $y = 8$.

$$\begin{aligned}
 8 &= -\frac{2}{1} + 1 + c \\
 c &= 9
 \end{aligned}$$

Hence, the equation of the curve is

$$y = -\frac{2}{x} + x + 9.$$

4 $\frac{dy}{dx} = 3x^2(x+1)$

$$\begin{aligned}
 y &= \int 3x^2(x+1) dx \\
 &= \int (3x^3 + 3x^2) dx \\
 &= \frac{3x^4}{4} + \frac{3x^3}{3} + c \\
 &= \frac{3x^4}{4} + x^3 + c
 \end{aligned}$$

Since the curve passes through the point (1, 2), thus $x = 1$ and $y = 2$.

$$\begin{aligned}
 2 &= \frac{3(1)^4}{4} + 1^3 + c \\
 c &= \frac{1}{4}
 \end{aligned}$$

Hence, the equation of the curve is

$$y = \frac{3x^4}{4} + x^3 + \frac{1}{4}.$$

5 $\frac{dy}{dx} = \frac{1}{(2x-1)^2}$

$$\begin{aligned}
 y &= \int \frac{1}{(2x-1)^2} dx \\
 &= \int (2x-1)^{-2} dx \\
 &= \frac{(2x-1)^{-1}}{-1(2)} + c \\
 &= -\frac{1}{2(2x-1)} + c
 \end{aligned}$$

Since the curve passes through the point (2, 1), thus $x = 2$ and $y = 1$.

$$\begin{aligned}
 1 &= -\frac{1}{2[2(2)-1]} + c \\
 1 &= -\frac{1}{6} + c \\
 c &= \frac{7}{6}
 \end{aligned}$$

Hence, the equation of the curve is

$$y = -\frac{1}{2(2x-1)} + \frac{7}{6}.$$

$$6 \quad \frac{dy}{dx} = 2x - 4$$

$$y = \int (2x - 4) dx$$

$$y = x^2 - 4x + c$$

For minimum value,

$$\frac{dy}{dx} = 0$$

$$2x - 4 = 0$$

$$x = 2$$

The minimum value is 3.

This means that the value of $y = 3$.

$$3 = 2^2 - 4(2) + c$$

$$c = 7$$

Hence, the equation of the curve is

$$y = x^2 - 4x + 7.$$

UPSKILL 3.3a

$$1 \text{ (a)} \quad \int_0^3 (2x^3 - 2x - 1) dx$$

$$= \left[\frac{2x^4}{4} - \frac{2x^2}{2} - x \right]_0^3$$

$$= \left[\frac{x^4}{2} - x^2 - x \right]_0^3$$

$$= \frac{3^4}{2} - 3^2 - 3 - 0$$

$$= \frac{57}{2}$$

$$(b) \quad \int_{-1}^1 \left(3x^2 + x + \frac{1}{x^2} \right) dx$$

$$= \left[x^3 + \frac{x^2}{2} - \frac{1}{x} \right]_{-1}^1$$

$$= 1^3 + \frac{1^2}{2} - \frac{1}{1} - \left[(-1)^3 + \frac{(-1)^2}{2} - \frac{1}{-1} \right]$$

$$= \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

$$(c) \quad \int_0^2 (3x-1)(x^2+1) dx$$

$$= \int_0^2 (3x^3 + 3x - x^2 - 1) dx$$

$$= \left[\frac{3x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - x \right]_0^2$$

$$= \frac{3(2)^4}{4} + \frac{3(2)^2}{2} - \frac{2^3}{3} - 2 - 0$$

$$= 13\frac{1}{3}$$

$$(d) \quad \int_1^2 \frac{3x^3 + 2}{x^3} dx$$

$$= \int_1^2 (3 + 2x^{-3}) dx$$

$$= \left[3x + \frac{2x^{-2}}{-2} \right]_1^2$$

$$= \left[3x - \frac{1}{x^2} \right]_1^2$$

$$= 3(2) - \frac{1}{2^2} - (3 - 1)$$

$$= \frac{15}{4}$$

$$(e) \quad \int_0^2 x(x+3)(x-3) dx$$

$$= \int_0^2 x(x^2 - 9) dx$$

$$= \int_0^2 (x^3 - 9x) dx$$

$$= \left[\frac{x^4}{4} - \frac{9x^2}{2} \right]_0^2$$

$$= \frac{2^4}{4} - \frac{9(2)^2}{2} - 0$$

$$= -14$$

$$\begin{aligned}
 \text{(f)} \quad & \int_{-1}^1 \left(2x - \frac{1}{x}\right)^2 dx \\
 &= \int_{-1}^1 (4x^2 - 4 + x^{-2}) dx \\
 &= \left[\frac{4x^3}{3} - 4x + \frac{x^{-1}}{-1} \right]_{-1}^1 \\
 &= \left[\frac{4x^3}{3} - 4x - \frac{1}{x} \right]_{-1}^1 \\
 &= \frac{4}{3} - 4 - 1 - \left(-\frac{4}{3} + 4 + 1 \right) \\
 &= -\frac{22}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad & \int_0^2 (2x+1)^3 dx \\
 &= \left[\frac{(2x+1)^4}{4(2)} \right]_0^2 \\
 &= \left[\frac{(2x+1)^4}{8} \right]_0^2 \\
 &= \frac{5^4}{8} - \frac{1}{8} \\
 &= 78
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad & \int_{-1}^1 \frac{1}{(3-x)^3} dx \\
 &= \int_{-1}^1 (3-x)^{-3} dx \\
 &= \left[\frac{(3-x)^{-2}}{-2(-1)} \right]_{-1}^1 \\
 &= \left[\frac{1}{2(3-x)^2} \right]_{-1}^1 \\
 &= \frac{1}{2(3-1)^2} - \frac{1}{2(3+1)^2} \\
 &= \frac{1}{8} - \frac{1}{32} \\
 &= \frac{3}{32}
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad & \int_0^2 x(x-2)^2 dx \\
 &= \int_0^2 x(x^2 - 4x + 4) dx \\
 &= \int_0^2 (x^3 - 4x^2 + 4x) dx \\
 &= \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2} \right]_0^2 \\
 &= \left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 \right]_0^2 \\
 &= \frac{2^4}{4} - \frac{4(2)^3}{3} + 2(2)^2 - 0 \\
 &= \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(j)} \quad & \int_0^2 \frac{(3x^3 - 2x^5 + 2)}{x^3} dx \\
 &= \int_0^2 (3 - 2x^2 + 2x^{-3}) dx \\
 &= \left[3x - \frac{2x^3}{3} + \frac{2x^{-2}}{-2} \right]_1^2 \\
 &= \left[3x - \frac{2x^3}{3} - \frac{1}{x^2} \right]_1^2 \\
 &= 3(2) - \frac{2}{3}(2)^3 - \frac{1}{2^2} - \left(3 - \frac{2}{3} - 1 \right) \\
 &= \frac{5}{12} - \frac{4}{3} \\
 &= -\frac{11}{12}
 \end{aligned}$$

$$\begin{aligned}
 \text{2 (a)} \quad & \int_2^6 5f(x) dx \\
 &= 5 \int_2^6 f(x) dx \\
 &= 5(4) \\
 &= 20
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_2^6 [5 - f(x)] dx \\
 &= \int_2^6 5 dx - \int_2^6 f(x) dx \\
 &= [5x]_2^6 - 4 \\
 &= 5(6-2) - 4 \\
 &= 16
 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \int_6^2 f(x) \, dx \\ &= -\int_2^6 f(x) \, dx \\ &= -4 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & \int_2^3 f(x) \, dx + \int_3^4 f(x) \, dx + \int_4^6 f(x) \, dx \\ &= \int_2^6 f(x) \, dx \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{3 (a)} \quad & \int_2^4 g(x) \, dx - \int_7^4 g(x) \, dx \\ &= \int_2^4 g(x) \, dx + \int_4^7 g(x) \, dx \\ &= 6 + 8 \\ &= 14 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \int_2^7 [g(x) + 2x] \, dx \\ &= \int_2^7 g(x) \, dx + \int_2^7 2x \, dx \\ &= 6 + 8 + [x^2]_2^7 \\ &= 14 + 7^2 - 2^2 \\ &= 59 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \int_4^2 3g(x) \, dx + \int_4^7 2g(x) \, dx \\ &= -3\int_2^4 g(x) \, dx + 2\int_4^7 g(x) \, dx \\ &= -3(6) + 2(8) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & \int_2^4 [g(x) + kx] \, dx = 10 \\ & \int_2^4 g(x) \, dx + \int_2^4 kx \, dx = 10 \\ & 6 + \frac{k}{2} [x^2]_2^4 = 10 \\ & \frac{k}{2} [4^2 - 2^2]_2^4 = 4 \\ & \frac{k}{2} \times 12 = 4 \\ & k = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{4} \quad & y = \frac{x}{x+4} \\ & \frac{dy}{dx} = \frac{(x+4)(1) - x(1)}{(x+4)^2} \\ & \frac{dy}{dx} = \frac{4}{(x+4)^2} \quad \text{[Shown]} \end{aligned}$$

$$\begin{aligned} y &= \int \frac{4}{(x+4)^2} \, dx \\ 4[y]_2^3 &= \int_2^3 \frac{16}{(x+4)^2} \, dx \\ 4 \left[\frac{x}{x+4} \right]_2^3 &= \int_2^3 \frac{16}{(x+4)^2} \, dx \\ \int_2^3 \frac{16}{(x+4)^2} \, dx &= 4 \left[\frac{x}{x+4} \right]_2^3 \\ &= 4 \left(\frac{3}{7} - \frac{2}{6} \right) \\ &= \frac{8}{21} \end{aligned}$$

$$\begin{aligned} \text{5} \quad & y = \frac{x}{5-x} \\ & \frac{dy}{dx} = \frac{(5-x)(1) - x(-1)}{(5-x)^2} \\ & \frac{dy}{dx} = \frac{5}{(5-x)^2} \quad \text{[Shown]} \\ y &= \int \frac{5}{(5-x)^2} \, dx \\ \frac{1}{10} [y]_1^3 &= \frac{1}{10} \int_1^3 \frac{5}{(5-x)^2} \, dx \\ \int_1^3 \frac{1}{2(5-x)^2} \, dx &= \frac{1}{10} \left[\frac{x}{5-x} \right]_1^3 \\ &= \frac{1}{10} \left(\frac{3}{2} - \frac{1}{4} \right) \\ &= \frac{1}{8} \end{aligned}$$

$$6 \quad \frac{dy}{dx} = 4f(x)$$

$$y = 4 \int f(x) dx$$

$$\frac{1}{4}y = \int f(x) dx$$

$$\int_1^2 f(x) dx = \frac{1}{4}[y]_1^2$$

$$= \frac{1}{4} \left[\frac{3x^2 - 1}{x} \right]_1^2$$

$$= \frac{1}{4} \left[\frac{3(2)^2 - 1}{2} - \frac{3(1)^2 - 1}{1} \right]$$

$$= \frac{1}{4} \left(\frac{11}{2} - 2 \right)$$

$$= \frac{7}{8}$$

$$7 \quad \frac{d}{dx} \left(\frac{x}{x^2 - 3} \right) = g(x)$$

$$\frac{x}{x^2 - 3} = \int g(x) dx$$

$$\left[\frac{x}{x^2 - 3} \right]_1^2 = \int_1^2 g(x) dx$$

$$\int_1^2 [2x - g(x)] dx = 2 \left[\frac{x^2}{2} \right]_1^2 - \left[\frac{x}{x^2 - 3} \right]_1^2$$

$$= (2^2 - 1^2) - \left(\frac{2}{1} - \frac{1}{-2} \right)$$

$$= 3 - \frac{5}{2}$$

$$= \frac{1}{2}$$

UPSKILL 3.3b

1 (a) Area of the shaded region

$$= \int_{-1}^4 y dx$$

$$= \int_{-1}^4 (1+x)(4-x) dx$$

$$= \int_{-1}^4 (4-x+4x-x^2) dx$$

$$= \int_{-1}^4 (4+3x-x^2) dx$$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-1}^4$$

$$= \left[16 + \frac{3}{2}(16) - \frac{64}{3} \right] - \left[-4 + \frac{3}{2} - \left(-\frac{1}{3} \right) \right]$$

$$= \frac{56}{3} - \left(-\frac{13}{6} \right)$$

$$= 20\frac{5}{6} \text{ units}^2$$

(b) Area of the shaded region

$$= \int_1^3 y dx$$

$$= \int_1^3 \left(x + \frac{2}{x^2} \right) dx$$

$$= \int_1^3 (x + 2x^{-2}) dx$$

$$= \left[\frac{x^2}{2} + 2 \left(\frac{x^{-1}}{-1} \right) \right]_1^3$$

$$= \left[\frac{x^2}{2} - 2 \left(\frac{1}{x} \right) \right]_1^3$$

$$= \frac{9}{2} - \frac{2}{3} - \left(\frac{1}{2} - 2 \right)$$

$$= 5\frac{1}{3} \text{ units}^2$$

(c) Area of the shaded region

$$= \int_{-1}^2 y dx$$

$$= \int_{-1}^2 (x-1)^2 dx$$

$$= \left[\frac{(x-1)^3}{3(1)} \right]_{-1}^2$$

$$= \left(\frac{1}{3} - \left(-\frac{8}{3} \right) \right)$$

$$= 3 \text{ units}^2$$

2 (a) Area of the shaded region

$$= \left| \int_{-2}^2 y dx \right|$$

$$= \left| \int_{-2}^2 (x+2)(x-2) dx \right|$$

$$= \left| \int_{-2}^2 (x^2 - 4) dx \right|$$

$$\begin{aligned}
&= \left| \left[\frac{x^3}{3} - 4x \right]_{-2}^2 \right| \\
&= \left| \frac{8}{3} - 8 - \left(\frac{-8}{3} + 8 \right) \right| \\
&= \left| -10\frac{2}{3} \right| \\
&= 10\frac{2}{3} \text{ units}^2
\end{aligned}$$

(b) Area of the shaded region

$$\begin{aligned}
&= \left| \int_0^3 y \, dx \right| \\
&= \left| \int_0^3 (x-3)^3 \, dx \right| \\
&= \left| \left[\frac{(x-3)^4}{4(1)} \right]_0^3 \right| \\
&= \left| 0 - \left(\frac{81}{4} \right) \right| \\
&= \frac{81}{4} \text{ units}^2
\end{aligned}$$

3 (a) $y = x(x+2)(x-1)$

$$\begin{aligned}
&= x(x^2 + x - 2) \\
&= x^3 + x^2 - 2x
\end{aligned}$$

Area of the shaded region

$$\begin{aligned}
&= \int_{-2}^0 (x^3 + x^2 - 2x) \, dx + \left| \int_0^1 (x^3 + x^2 - 2x) \, dx \right| \\
&= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{-2}^0 + \left| \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_0^1 \right| \\
&= 0 - \left(\frac{16}{4} - \frac{8}{3} - 4 \right) + \left| \left(\frac{1}{4} + \frac{1}{3} - 1 - 0 \right) \right| \\
&= \frac{8}{3} + \left| -\frac{5}{12} \right| \\
&= 3\frac{1}{12} \text{ units}^2
\end{aligned}$$

(b) $y = x(1+x)(4-x)$

$$\begin{aligned}
&y = x(4 + 3x - x^2) \\
&y = 4x + 3x^2 - x^3
\end{aligned}$$

Area of the shaded region

$$\begin{aligned}
&= \left| \int_{-1}^0 (4x + 3x^2 - x^3) \, dx \right| + \int_0^4 (4x + 3x^2 - x^3) \, dx \\
&= \left| \left[2x^2 + x^3 - \frac{x^4}{4} \right]_{-1}^0 \right| + \left[2x^2 + x^3 - \frac{x^4}{4} \right]_0^4 \\
&= \left| 0 - \left(2 - 1 - \frac{1}{4} \right) \right| + 2(16) + 64 - \frac{4^4}{4} - 0 \\
&= \left| -\frac{3}{4} \right| + 32 \\
&= 32\frac{3}{4} \text{ units}^2
\end{aligned}$$

(c) Area of the shaded region

$$\begin{aligned}
&= \int_0^3 (3+2x-x^2) \, dx + \left| \int_3^4 (3+2x-x^2) \, dx \right| \\
&= \left[3x + x^2 - \frac{x^3}{3} \right]_0^3 + \left| \left[3x + x^2 - \frac{x^3}{3} \right]_3^4 \right| \\
&= 9 + 9 - 9 + \left| 3(4) + 4^2 - \frac{64}{3} - \left[3(3) + 3^2 - \frac{27}{3} \right] \right| \\
&= 9 + \left| \frac{20}{3} - 9 \right| \\
&= 9 + \frac{7}{3} \\
&= 11\frac{1}{3} \text{ units}^2
\end{aligned}$$

4 (a) Area of the shaded region

$$\begin{aligned}
&= \int_{-1}^3 x \, dy \\
&= \int_{-1}^3 (3y^2 + 2) \, dy \\
&= \left[\frac{3y^3}{3} + 2y \right]_{-1}^3 \\
&= \left[y^3 + 2y \right]_{-1}^3 \\
&= 27 + 6 - (-1 - 2) \\
&= 36 \text{ units}^2
\end{aligned}$$

(b) Area of the shaded region

$$\begin{aligned}
 &= \int_0^3 x \, dy \\
 &= \int_0^3 y^2(3-y) \, dy \\
 &= \int_0^3 (3y^2 - y^3) \, dy \\
 &= \left[y^3 - \frac{y^4}{4} \right]_0^3 \\
 &= 3^3 - \frac{3^4}{4} \\
 &= 6\frac{3}{4} \text{ units}^2
 \end{aligned}$$

5 (a) Area of the shaded region

$$\begin{aligned}
 &= \left| \int_{-1}^3 x \, dy \right| \\
 &= \left| \int_{-1}^3 (y^2 - 2y - 3) \, dy \right| \\
 &= \left| \left[\frac{y^3}{3} - y^2 - 3y \right]_{-1}^3 \right| \\
 &= \left| 9 - 9 - 9 - \left(-\frac{1}{3} - 1 + 3 \right) \right| \\
 &= \left| -9 - \frac{5}{3} \right| \\
 &= \left| -10\frac{2}{3} \right| \\
 &= 10\frac{2}{3} \text{ units}^2
 \end{aligned}$$

(b) $x = y(y+1)(y-1)$

$$\begin{aligned}
 x &= y(y^2 - 1) \\
 x &= (y^3 - y)
 \end{aligned}$$

Area of the shaded region

$$\begin{aligned}
 &= \int_{-1}^0 (y^3 - y) \, dy + \left| \int_0^1 (y^3 - y) \, dy \right| \\
 &= \left[\frac{y^4}{4} - \frac{y^2}{2} \right]_{-1}^0 + \left| \left[\frac{y^4}{4} - \frac{y^2}{2} \right]_0^1 \right| \\
 &= 0 - \left(\frac{1}{4} - \frac{1}{2} \right) + \left| \frac{1}{4} - \frac{1}{2} - 0 \right| \\
 &= \frac{1}{4} + \frac{1}{4} \\
 &= \frac{1}{2} \text{ unit}^2
 \end{aligned}$$

6 (a) (i) $y = -x + 4$... (1)

$$y = 4 + 3x - x^2 \quad \dots (2)$$

Substitute (1) into (2) :

$$-x + 4 = 4 + 3x - x^2$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0 \text{ or } x = 4$$

At the turning point, $x = 0$,

$$y = -0 + 4 = 4$$

At the turning point, $x = 4$,

$$y = -4 + 4 = 0$$

Thus, the coordinates of the points P and Q are $(0, 4)$ and $(4, 0)$ respectively.

(ii) Area of the shaded region

= (Area under the curve from $x = 0$ to $x = 4$) - (Area of the triangle from $x = 0$ to $x = 4$)

$$= \int_0^4 (4 + 3x - x^2) \, dx - \frac{1}{2} \times 4 \times 4$$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^4 - 8$$

$$= 4(4) + \frac{3}{2}(4)^2 - \frac{4^3}{3} - 8$$

$$= \frac{56}{3} - 8$$

$$= 10\frac{2}{3} \text{ units}^2$$

(b) (i) $y = 21$... (1)

$$y = 10x - x^2 \quad \dots (2)$$

$$21 = 10x - x^2$$

$$x^2 - 10x + 21 = 0$$

$$(x-3)(x-7) = 0$$

$$x = 3 \text{ or } x = 7$$

Thus, the coordinates of the points P and Q are $(3, 21)$ and $(7, 21)$ respectively.

(ii) Area of the shaded region

= (Area under the curve from $x = 3$ to $x = 7$) - (Area of the rectangle from $x = 3$ to $x = 7$)

$$= \int_3^7 (10x - x^2) \, dx - (4 \times 21)$$

$$= \left[5x^2 - \frac{x^3}{3} \right]_3^7 - 84$$

$$= \left[5(7)^2 - \frac{7^3}{3} - \left(5(3)^2 - \frac{3^3}{3} \right) \right] - 84$$

$$= \left(\frac{392}{3} - 36 \right) - 84$$

$$= 10 \frac{2}{3} \text{ units}^2$$

(c) (i) $x = 9 \dots (1)$

$$y^2 = 4x \dots (2)$$

Substitute (1) into (2) :

$$y^2 = 4(9) = 36$$

$$y = \pm 6$$

Thus, the coordinates of the points P and Q are $(9, 6)$ and $(9, -6)$ respectively.

(ii) Area of the shaded region

= (Area of the rectangle from $y = -6$ to $y = 6$) - (Area between the curve and the y -axis from $y = -6$ to $y = 6$)

$$= 12 \times 9 - \int_{-6}^6 \frac{y^2}{4} dy$$

$$= 108 - \left[\frac{y^3}{12} \right]_{-6}^6$$

$$= 108 - \left[\frac{6^3}{12} - \left(\frac{-6^3}{12} \right) \right]$$

$$= 108 - (18 + 18)$$

$$= 72 \text{ units}^2$$

(d) (i) $y = 4x + 1 \dots (1)$

$$y = x^2 + 5 \dots (2)$$

Substitute (2) into (1) :

$$x^2 + 5 = 4x + 1$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$x = 2$$

From (1) : $y = 4(2) + 1 = 9$

Thus, the coordinates of point Q are $(2, 9)$.

$$y = 4x + 1$$

At the y -axis, $x = 0$.

$$y = 4(0) + 1 = 1$$

Thus, the coordinates of point P are $(0, 1)$.

(ii) Area of the shaded region

= (Area under the curve from $x = 0$ to $x = 2$) - (Area of trapezium from $x = 0$ to $x = 2$)

$$= \int_0^2 (x^2 + 5) - \frac{1}{2}(1 + 9)(2)$$

$$= \left[\frac{x^3}{3} + 5x \right]_0^2 - 10$$

$$= \frac{8}{3} + 10 - 10$$

$$= \frac{8}{3} \text{ units}^2$$

7 (a) (i) $y = x + 3 \dots (1)$

$$y = 2x^2 \dots (2)$$

Substitute (2) into (1) :

$$2x^2 = x + 3$$

$$2x^2 - x - 3 = 0$$

$$x = \frac{3}{2} \text{ or } x = -1$$

$x = \frac{3}{2}$ is not accepted.

$$\therefore x = -1$$

From (1) : $y = -1 + 3 = 2$

Thus, the coordinates of point Q are $(-1, 2)$.

$$y = x + 3$$

At the x -axis, $y = 0$.

$$y = x + 3$$

$$0 = x + 3$$

$$x = -3$$

Thus, the coordinates of point P are $(-3, 0)$.

(ii) Area of the shaded region

= (Area of the triangle from $x = -3$ to $x = -1$) + (Area under the curve from $x = -1$ to $x = 0$)

$$= \frac{1}{2} \times 2 \times 2 + \int_{-1}^0 2x^2 dx$$

$$= 2 + \left[\frac{2x^3}{3} \right]_{-1}^0$$

$$= 2 + \frac{2}{3} [0 - (-1)]$$

$$= 2 + \frac{2}{3}$$

$$= 2 \frac{2}{3} \text{ units}^2$$

(b) (i) $y = -x + 6 \dots (1)$

$$2x = (y - 2)^2 \dots (2)$$

Substitute (1) into (2) :

$$2x = (-x + 6 - 2)^2$$

$$\begin{aligned}
2x &= (-x+4)^2 \\
2x &= x^2 - 8x + 16 \\
x^2 - 10x + 16 &= 0 \\
(x-2)(x-8) &= 0 \\
x &= 2 \text{ or } x = 8 \\
x = 8 &\text{ is not accepted.} \\
\therefore x &= 2
\end{aligned}$$

$$\text{From (1): } y = -2 + 6 = 4$$

Thus, the coordinates of point Q are $(2, 4)$.

$$x + y = 6$$

At the y -axis, $x = 0$.

$$0 + y = 6$$

$$y = 6$$

Thus, the coordinates of point P are $(0, 6)$.

- (ii) Area of the shaded region
= (Area between the curve and the y -axis from $y = 2$ to $y = 4$) + (Area of the triangle from $y = 4$ to $y = 6$)

$$\begin{aligned}
&= \int_2^4 \frac{(y-2)^2}{2} dy + \frac{1}{2} \times 2 \times 2 \\
&= \left[\frac{(y-2)^3}{2(3)(1)} \right]_2^4 + 2 \\
&= \frac{1}{6} (2^3 - 0) + 2 \\
&= \frac{10}{3} \text{ units}^2
\end{aligned}$$

8 Area of the shaded region = 81

$$\begin{aligned}
\int_0^k y \, dx &= 81 \\
\int_0^k 4x^3 \, dx &= 81 \\
\left[x^4 \right]_0^k &= 81 \\
k^4 - 0 &= 81 \\
k &= 3
\end{aligned}$$

9 Area of the shaded region = $\frac{1}{3}$

$$\begin{aligned}
\int_0^k x \, dy &= \frac{1}{3} \\
\int_0^k \frac{y^2}{8} \, dx &= \frac{1}{3} \\
\left[\frac{y^3}{24} \right]_0^k &= \frac{1}{3}
\end{aligned}$$

$$k^3 = \frac{24}{3}$$

$$k^3 = 8$$

$$k = 2$$

10 Area of the shaded region

$$\begin{aligned}
&= \int_0^3 \left[(-2x^2 + 6x) - (x^2 - 3x) \right] dx \\
&= \int_0^3 (-3x^2 + 9x) dx \\
&= \left[-x^3 + \frac{9x^2}{2} \right]_0^3 \\
&= \left(-3^3 + \frac{9}{2} \times 3^2 - 0 \right) \\
&= 13 \frac{1}{2} \text{ units}^2
\end{aligned}$$

11 Area of the shaded region

$$\begin{aligned}
&= \int_{-1}^1 \left[(x^3 - x) - (x^2 - 1) \right] dx \\
&= \int_{-1}^1 (x^3 - x^2 - x + 1) dx \\
&= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 + x \right]_{-1}^1 \\
&= \frac{1}{4} - \frac{1}{3} - 1 + 1 - \left(\frac{1}{4} + \frac{1}{3} - 1 - 1 \right) \\
&= -\frac{1}{12} - \left(-\frac{17}{12} \right) \\
&= \frac{4}{3} \text{ units}^2
\end{aligned}$$

UPSKILL 3.3c

1 (a) Generated volume

$$\begin{aligned}
&= \pi \int_1^4 y^2 \, dx \\
&= \pi \int_1^4 \left(\frac{4}{x} \right)^2 dx \\
&= \pi \int_1^4 \frac{16}{x^2} dx \\
&= \pi \int_1^4 16x^{-2} dx \\
&= \pi \left[\frac{16x^{-1}}{-1} \right]_1^4 \\
&= \pi \left[-\frac{16}{x} \right]_1^4
\end{aligned}$$

$$= \pi \left[-\frac{16}{4} - (-16) \right]$$

$$= 12\pi \text{ units}^3$$

(b) Generated volume

$$= \pi \int_1^3 \left(x + \frac{2}{x} \right)^2 dx$$

$$= \pi \int_1^3 (x^2 + 4 + 4x^{-2}) dx$$

$$= \pi \left[\frac{x^3}{3} + 4x + \frac{4x^{-1}}{-1} \right]_1^3$$

$$= \pi \left[\frac{x^3}{3} + 4x - \frac{4}{x} \right]_1^3$$

$$= \pi \left[9 + 12 - \frac{4}{3} - \left(\frac{1}{3} + 4 - 4 \right) \right]$$

$$= \pi \left(\frac{59}{3} - \frac{1}{3} \right)$$

$$= \frac{58}{3} \pi \text{ units}^2$$

(c) Generated volume

$$= \pi \int_{-1}^2 (4 - x^2)^2 dx$$

$$= \pi \int_{-1}^2 (16 - 8x^2 + x^4) dx$$

$$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-1}^2$$

$$= \pi \left[32 - \frac{64}{3} + \frac{32}{5} - \left(-16 + \frac{8}{3} - \frac{1}{5} \right) \right]$$

$$= \frac{153}{5} \pi \text{ units}^2$$

(d) Generated volume

$$= \pi \int_0^2 (x^2 - 3x)^2 dx$$

$$= \pi \int_0^2 (x^4 - 6x^3 + 9x^2) dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{6x^4}{4} + \frac{9x^3}{3} \right]_0^2$$

$$= \pi \left[\frac{x^5}{5} - \frac{3x^4}{2} + 3x^3 \right]_0^2$$

$$= \pi \left[\frac{32}{5} - \frac{3(16)}{2} + 3(8) - 0 \right]$$

$$= \frac{32}{5} \text{ units}^2$$

(e) Generated volume

$$= \pi \int_0^3 [(x-2)^2]^2 dx$$

$$= \pi \left[\frac{(x-2)^5}{5(1)} \right]_0^3$$

$$= \pi \left[\frac{1}{5} - \left(-\frac{32}{5} \right) \right]$$

$$= \frac{33}{5} \pi \text{ units}^2$$

(f) Generated volume

$$= \pi \int_{\frac{1}{2}}^1 y^2 dx$$

$$= \pi \int_{\frac{1}{2}}^1 \left(4x - \frac{1}{x} \right)^2 dx$$

$$= \pi \int_{\frac{1}{2}}^1 (16x^2 - 8 + x^{-2}) dx$$

$$= \pi \left[\frac{16x^3}{3} - 8x + \frac{x^{-1}}{-1} \right]_{\frac{1}{2}}^1$$

$$= \pi \left[\frac{16x^3}{3} - 8x - \frac{1}{x} \right]_{\frac{1}{2}}^1$$

$$= \pi \left[\frac{16}{3} - 8 - 1 - \left[\left(\frac{16}{3} \right) \left(\frac{1}{8} \right) - \frac{8}{2} - 2 \right] \right]$$

$$= \pi \left[-\frac{11}{3} - \left(-\frac{16}{3} \right) \right]$$

$$= \frac{5}{3} \pi \text{ units}^2$$

2 (a) Generated volume

$$= \pi \int_0^2 x^2 dy$$

$$= \pi \int_0^2 \left(\frac{y^2}{4} \right)^2 dy$$

$$= \pi \int_0^2 \frac{y^4}{16} dy$$

$$= \pi \left[\frac{y^5}{16(5)} \right]_0^2$$

$$= \pi \left(\frac{32}{80} - 0 \right)$$

$$= \frac{2}{5} \pi \text{ units}^3$$

(b) Generated volume

$$\begin{aligned} &= \pi \int_0^1 x^2 dy \\ &= \pi \int_0^1 [(y-1)^2]^2 dy \\ &= \pi \int_0^1 (y-1)^4 dy \\ &= \pi \left[\frac{(y-1)^5}{5(1)} \right]_0^1 \\ &= \pi \left[0 - \left(-\frac{1}{5} \right) \right] \\ &= \frac{1}{5} \pi \text{ units}^3 \end{aligned}$$

(c) Generated volume

$$\begin{aligned} &= \pi \int_0^1 x^2 dy \\ &= \pi \int_0^1 (y^2 - 2y)^2 dy \\ &= \pi \int_0^1 (y^4 - 4y^3 + 4y^2) dy \\ &= \pi \left[\frac{y^5}{5} - y^4 + \frac{4y^3}{3} \right]_0^1 \\ &= \frac{32}{5} - 16 + \frac{32}{3} - 0 \\ &= \frac{16}{15} \pi \text{ units}^3 \end{aligned}$$

(d) Generated volume

$$\begin{aligned} &= \pi \int_1^2 x^2 dy \\ &= \pi \int_1^2 \left(\frac{y^2}{6} \right)^2 dy \\ &= \pi \int_1^2 \left(\frac{y^4}{36} \right) dy \\ &= \pi \left[\frac{y^5}{180} \right]_1^2 \\ &= \pi \left[\frac{2^5}{180} - \frac{1}{180} \right]_1^2 \\ &= \frac{31}{180} \pi \text{ units}^3 \end{aligned}$$

(e) Generated volume

$$\begin{aligned} &= \pi \int_0^9 (9-y) dy \\ &= \pi \left[9y - \frac{y^2}{2} \right]_0^9 \\ &= \pi \left(81 - \frac{81}{2} \right) \\ &= \frac{81}{2} \pi \text{ units}^3 \end{aligned}$$

(f) Generated volume

$$\begin{aligned} &= \pi \int_0^3 x^2 dy \\ &= \pi \int_0^3 [-(y-3)^2]^2 dy \\ &= \pi \int_0^3 [(y-3)^4] dy \\ &= \pi \left[\frac{(y-3)^5}{5(1)} \right]_0^3 \\ &= \pi \left[0 - \left(-\frac{243}{5} \right) \right] \\ &= \frac{243}{5} \pi \text{ units}^3 \end{aligned}$$

3 Generated volume = π

$$\begin{aligned} \pi \int_1^k y^2 dx &= \pi \\ \int_1^k \left(\frac{-\sqrt{2x+1}}{2} \right)^2 dx &= 1 \\ \int_1^k \left(\frac{2x+1}{4} \right) dx &= 1 \\ \int_1^k (2x+1) dx &= 4 \\ [x^2 + x]_1^k &= 4 \\ k^2 + k - (1+1) &= 4 \\ k^2 + k - 6 &= 0 \\ (k+3)(k-2) &= 0 \\ k = -2 \text{ or } k = 2 \\ k = -2 \text{ is not accepted because } k &\text{ has} \\ \text{to be positive.} \\ \therefore k = 2 \end{aligned}$$

4 Generated volume = $\frac{9}{2} \pi$

$$\pi \int_k^4 x^2 dy = \frac{9}{2} \pi$$

$$\int_k^4 (4-y) dy = \frac{9}{2}$$

$$\left[4y - \frac{y^2}{2} \right]_k^4 = \frac{9}{2}$$

$$4(4) - \frac{4^2}{2} - \left(4k - \frac{k^2}{2} \right) = \frac{9}{2}$$

$$8 - 4k + \frac{k^2}{2} - \frac{9}{2} = 0$$

$$16 - 8k + k^2 - 9 = 0$$

$$k^2 - 8k + 7 = 0$$

$$(k-7)(k-1) = 0$$

$$k = 7 \text{ or } k = 1$$

$k = 7$ is not accepted because $k < 4$.

$$\therefore k = 1$$

5 Generated volume = 2π

$$\pi \int_2^k x^2 dy = 2\pi$$

$$\int_2^k (y-2) dy = 2$$

$$\left[\frac{y^2}{2} - 2y \right]_2^k = 2$$

$$\frac{k^2}{2} - 2k - (2-4) = 2$$

$$\frac{k^2}{2} - 2k + 2 = 2$$

$$k^2 - 4k = 0$$

$$k(k-4) = 0$$

$$k = 0 \text{ or } k = 4$$

$k = 0$ is not accepted.

$$\therefore k = 4$$

6 $y = \frac{8}{x}$

At the turning point, $y = \frac{4}{3}$,

$$\frac{4}{3} = \frac{8}{x}$$

$$4x = 24$$

$$x = 6$$

Generated volume

= (Volume generated by the curve from $x = 2$ to $x = 6$) - (Volume generated by $y = \frac{4}{3}$ from $x = 2$ to $x = 6$)

$$= \pi \int_2^6 \left(\frac{8}{x} \right)^2 dx - (\text{Volume of cylinder}$$

generated with a radius of $\frac{4}{3}$ units and a height of 4 units)

$$= \pi \int_2^6 64x^{-2} dx - \pi r^2 h$$

$$= \pi \left[\frac{64x^{-1}}{-1} \right]_2^6 - \pi \left(\frac{4}{3} \right)^2 (4)$$

$$= \pi \left[-\frac{64}{x} \right]_2^6 + \frac{64}{9} \pi$$

$$= \pi \left[-\frac{64}{6} - \left(-\frac{64}{2} \right) \right] - \frac{64}{9} \pi$$

$$= \frac{64}{3} \pi - \frac{64}{9} \pi$$

$$= \frac{128}{9} \pi \text{ units}^3$$

7 (a) $y = 6x - x^2$

$$\frac{dy}{dx} = 6 - 2x$$

At maximum point,

$$\frac{dy}{dx} = 0$$

$$6 - 2x = 0$$

$$x = 3$$

At the turning point $x = 3$,

$$y = 6(3) - 3^2 = 9$$

Thus, the coordinates of the maximum point are $Q(3, 9)$.

(b) Generated volume

= (Volume of the cylinder generated by the straight line $y = 9$ from $x = 0$ to $x = 3$) - (Volume generated by the curve from $x = 0$ to $x = 3$)

$$= \pi r^2 h - \pi \int_0^3 (6x - x^2)^2 dx$$

$$= \pi (9)^2 (3) - \pi \int_0^3 (36x^2 - 12x^3 + x^4) dx$$

$$= 243\pi - \pi \left[\frac{36x^3}{3} - \frac{12x^4}{4} + \frac{x^5}{5} \right]_0^3$$

$$\begin{aligned}
&= 243\pi - \pi \left[\frac{36x^3}{3} - \frac{12x^4}{4} + \frac{x^5}{5} \right]_0^3 \\
&= 243\pi - \pi \left(12(3)^3 - 3(3)^4 + \frac{3^5}{5} - 0 \right) \\
&= 243\pi - \frac{648}{5}\pi \\
&= \frac{567}{5}\pi \text{ units}^3
\end{aligned}$$

8 $y = \frac{4}{x} \dots (1)$

$y = -2x + 6 \dots (2)$

Substitute (1) into (2) :

$$\frac{4}{x} = -2x + 6$$

$$4 = -2x^2 + 6x$$

$$2x^2 - 6x + 4 = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$x = 1 \text{ or } x = 2$$

At the turning point, $x = 1$, $y = \frac{4}{1} = 4$

At the turning point, $x = 2$, $y = \frac{4}{2} = 2$

Thus, the points of intersection between the straight line and the curve are (1, 4) and (2, 2).

The straight line $y = -2x + 6$ intersects the x -axis at (3, 0).

(Volume generated by the curve from $x = 1$ to $x = 2$) + (Volume of the cone generated by the straight line from $x = 2$ to $x = 3$)

$$\begin{aligned}
&= \pi \int_1^2 \left(\frac{4}{x} \right)^2 dx + \frac{1}{3} \pi r^2 h \\
&= \pi \int_1^2 (16x^{-2}) dx + \frac{1}{3} \pi (2)^2 (3-2) \\
&= \pi \left[\frac{16x^{-1}}{-1} \right]_1^2 + \frac{4}{3} \pi \\
&= \pi \left[-\frac{16}{x} \right]_1^2 + \frac{4}{3} \pi \\
&= \left(-\frac{16}{2} + \frac{16}{1} \right) \pi + \frac{4}{3} \pi \\
&= 8\pi + \frac{4}{3} \pi \\
&= \frac{28}{3} \pi \text{ units}^3
\end{aligned}$$

9 Generated volume

= (Volume of the cone generated by the straight line from $x = 0$ to $x = 2$) - (Volume generated by the curve from $x = 1$ to $x = 2$)

$$\begin{aligned}
&= \frac{1}{3} \pi r^2 h - \pi \int_1^2 y^2 dx \\
&= \frac{1}{3} \pi (1)^2 (2) - \pi \int_1^2 (x-1) dx \\
&= \frac{2}{3} \pi - \pi \left[\frac{x^2}{2} - x \right]_1^2 \\
&= \frac{2}{3} \pi - \pi \left[\frac{2^2}{2} - 2 - \left(\frac{1}{2} - 1 \right) \right] \\
&= \frac{2}{3} \pi - \frac{1}{2} \pi \\
&= \frac{1}{6} \pi \text{ units}^3
\end{aligned}$$

10 (a) $y = 3x + 6 \dots (1)$

$y = 16 - x^2 \dots (2)$

$$3x + 6 = 16 - x^2$$

$$x^2 + 3x - 10 = 0$$

$$(x-2)(x+5) = 0$$

$$x = 2 \text{ or } x = -5$$

$x = -5$ is not accepted.

$$\therefore x = 2$$

At the turning point, $x = 2$, $y = 3(2) + 6 = 12$

Thus, the coordinates of point P are (2, 12).

(b) The straight line $y = 3x + 6$ intersects the y -axis at (0, 6).

The curve $y = 16 - x^2$ intersects the x -axis at (4, 0).

Area of the shaded region

= (Area under the straight line from $x = 0$ to $x = 2$) + (Area under the curve from $x = 2$ to $x = 4$)

$$\begin{aligned}
&= \frac{1}{2} (6+12)(2) + \int_2^4 (16-x^2) dx \\
&= 18 + \left[16x - \frac{x^3}{3} \right]_2^4 \\
&= 18 + \left[16(4) - \frac{4^3}{3} - \left(16(2) - \frac{2^3}{3} \right) \right] \\
&= 18 + \frac{128}{3} - \frac{88}{3} \\
&= \frac{94}{3} \text{ units}^2
\end{aligned}$$

(c) The y-intercept of the curve

$$y = 16 - x^2 \text{ is } 16.$$

Generated volume

$$\begin{aligned} &= \pi \int_{12}^{16} x^2 dy \\ &= \int_{12}^{16} \pi(16-y) dy \\ &= \pi \left[16y - \frac{y^2}{2} \right]_{12}^{16} \\ &= \pi \left[16(16) - \frac{16^2}{2} \right] - \pi \left[16(12) - \frac{12^2}{2} \right] \\ &= 128\pi - 120\pi \\ &= 8\pi \text{ units}^3 \end{aligned}$$

UPSKILL 3.4

1 (a) $\frac{dT}{dt} = \frac{1}{2}(14-t)$

At the turning point, T is a maximum,

$$\frac{dT}{dt} = 0$$

$$\frac{1}{2}(14-t) = 0$$

$$t = 14$$

$$\frac{d^2T}{dt^2} = -\frac{1}{2} \text{ (Negative)}$$

Thus, T is a maximum.

(b) $\frac{dT}{dt} = \frac{1}{2}(14-t)$

$$T = \frac{1}{2} \int (14-t) dt$$

$$T = \frac{1}{2} \left(14t - \frac{t^2}{2} \right) + c$$

Given that $T = 25$ at the turning point where $t = 8$,

$$25 = \frac{1}{2} \left(14(8) - \frac{8^2}{2} \right) + c$$

$$25 = 40 + c$$

$$c = -15$$

$$\text{Thus, } T = \frac{1}{2} \left(14t - \frac{t^2}{2} \right) - 15$$

At the turning point where $t = 14$, the maximum value of T

$$= \frac{1}{2} \left[14(14) - \frac{14^2}{2} \right] - 15$$

$$= 34$$

2 (a) Sphere with a radius of 7 cm

(b) Volume of sphere

$$\begin{aligned} &= \pi \int_{-7}^7 x^2 dy \\ &= \pi \int_{-7}^7 (49 - y^2) dy \\ &= \pi \left[49y - \frac{y^3}{3} \right]_{-7}^7 \\ &= \pi \left[49(7) - \frac{7^3}{3} - \left((49)(-7) - \frac{(-7)^3}{3} \right) \right] \\ &= \pi \left(\frac{686}{3} - \left(-\frac{686}{3} \right) \right) \\ &= 457\frac{1}{3}\pi \text{ cm}^3 \end{aligned}$$

Summative Practice 3

1 $\int \left(t + \frac{1}{t} \right) \left(t - \frac{1}{t} \right) dt$

$$= \int \left(t^2 - \frac{1}{t^2} \right) dt$$

$$= \int (t^2 - t^{-2}) dt$$

$$= \frac{t^3}{3} - \frac{t^{-1}}{-1} + c$$

$$= \frac{t^3}{3} + \frac{1}{t} + c$$

2 $\int \frac{(1+x)(1-x)}{x^4} dx$

$$= \int \frac{1-x^2}{x^4} dx$$

$$= \int (x^{-4} - x^{-2}) dx$$

$$= \frac{x^{-3}}{-3} - \frac{x^{-1}}{-1} + c$$

$$= -\frac{1}{3x^3} + \frac{1}{x} + c$$

3 $\int \frac{10}{(5-2y)^3} dy$

$$= \int 10(5-2y)^{-3} dy$$

$$= 10 \left[\frac{(5-2y)^{-2}}{-2(-2)} \right] + c$$

$$= \frac{5}{2(5-2y)^2} + c$$

$$\begin{aligned}
 4 \int \frac{3}{(1+2x)^4} dx &= \int 3(1+2x)^{-4} dx \\
 &= \left[\frac{3(1+2x)^{-3}}{-3(2)} \right] + c \\
 &= -\frac{1}{2}(1+2x)^{-3} + c
 \end{aligned}$$

But the given integral is $k(1+2x)^n + c$.

By comparison, $k = -\frac{1}{2}$ and $n = -3$.

$$\begin{aligned}
 5 \frac{d}{dx}[f(x)] &= 4g(x) \\
 f(x) &= 4 \int g(x) dx \\
 \int g(x) dx &= \frac{1}{4}[f(x)] + c
 \end{aligned}$$

$$\begin{aligned}
 6 \text{ (a)} \quad \frac{dy}{dx} &= kx - 5 \\
 y &= \int (kx - 5) dx \\
 y &= \frac{kx^2}{2} - 5x + c \\
 \text{The curve passes through the point } & (1, -1), \text{ thus } x = 1 \text{ and } y = -1. \\
 -1 &= \frac{k}{2} - 5 + c \\
 -2 &= k - 10 + 2c \\
 k + 2c &= 8 \dots (1)
 \end{aligned}$$

The curve passes through the point (0, 3), thus $x = 0$ and $y = 3$.

$$\begin{aligned}
 y &= \frac{kx^2}{2} - 5x + c \\
 3 &= 0 - 0 + c \\
 c &= 3
 \end{aligned}$$

From (1) : $k + 2(3) = 8 \Rightarrow k = 2$

$$\begin{aligned}
 \text{(b)} \quad \frac{dy}{dx} &= 2x - 5 \\
 y &= \int (2x - 5) dx \\
 y &= x^2 - 5x + h \\
 \text{The curve passes through the point } & (0, 3), \text{ thus } x = 0 \text{ and } y = 3. \\
 3 &= 0 - 0 + h \\
 h &= 3 \\
 \text{Thus, } y &= x^2 - 5x + 3
 \end{aligned}$$

$$\begin{aligned}
 7 \text{ (a)} \quad \frac{dy}{dx} &= 2x + k \\
 y &= \int 2x + k \\
 y &= x^2 + kx + c
 \end{aligned}$$

The turning point is (1, 4).

$$\begin{aligned}
 \frac{dy}{dx} &= 2(1) + k = 0 \\
 k &= -2
 \end{aligned}$$

(b) The curve passes through the point (1, 4), thus $x = 1$ and $y = 4$.

$$\begin{aligned}
 y &= x^2 - 2x + c \\
 4 &= 1^2 - 2(1) + c \\
 c &= 5
 \end{aligned}$$

Thus, the curve intersects the y-axis at the point (0, 5).

$$8 \text{ Gradient of normal} = \frac{1}{3-2x}$$

Gradient of tangent = $-(3-2x)$.

$$\begin{aligned}
 \frac{dy}{dx} &= -3 + 2x \\
 y &= \int (-3 + 2x) dx
 \end{aligned}$$

$$y = -3x + x^2 + c$$

The curve passes through the point (3, 4), thus $x = 3$ and $y = 4$.

$$\begin{aligned}
 4 &= -3(3) + 3^2 + c \\
 c &= 4
 \end{aligned}$$

Hence, the equation of the curve is

$$y = -3x + x^2 + 4.$$

9 (a) Gradient of tangent

$$\begin{aligned}
 &= \frac{2-0}{4-2} \\
 &= 1
 \end{aligned}$$

$$\text{(b)} \quad \frac{dy}{dx} = 1$$

$$\frac{1}{2}x + p = 1$$

$$\frac{1}{2}(4) + p = 1$$

$$p = -1$$

$$\text{(c)} \quad \frac{dy}{dx} = \frac{1}{2}x - 1$$

$$y = \int \left(\frac{1}{2}x - 1 \right) dx$$

$$y = \frac{1}{2} \left(\frac{x^2}{2} \right) - x + c$$

$$y = \frac{1}{4}x^2 - x + c$$

The curve passes through the point (4, 2), thus $x = 4$ and $y = 2$.

$$2 = \frac{1}{4}(4)^2 - 4 + c$$

$$2 = 4 - 4 + c$$

$$c = 2$$

Hence, the equation of the curve is

$$y = \frac{1}{4}x^2 - x + 2.$$

10 (a) The equation of the normal is

$$x + 2y - 5 = 0.$$

$$x + 2y - 5 = 0$$

$$2y = -x + 5$$

$$y = -\frac{1}{2}x + \frac{5}{2}$$

Since the gradient of the normal is $-\frac{1}{2}$,

thus the gradient of the tangent is 2.

$$\frac{dy}{dx} = 2$$

$$\frac{k}{x^2} + 3 = 2$$

$$\frac{k}{1^2} + 3 = 2$$

$$k = -1$$

$$(b) \frac{dy}{dx} = -\frac{1}{x^2} + 3$$

$$y = \int \left(-\frac{1}{x^2} + 3 \right) dx$$

$$y = \int \left(-x^{-2} + 3 \right) dx$$

$$y = \frac{-x^{-1}}{-1} + 3x + c$$

$$y = \frac{1}{x} + 3x + c$$

The curve passes through the point (1, 2), thus $x = 1$ and $y = 2$.

$$y = \frac{1}{x} + 3x + c$$

$$2 = \frac{1}{1} + 3(1) + c$$

$$c = -2$$

Hence, the equation of the curve is

$$y = \frac{1}{x} + 3x - 2.$$

$$11 (a) \frac{dy}{dx} = 2x + 4$$

$$y = \int (2x + 4) dx$$

$$y = x^2 + 4x + c$$

Given $y = 7$ at the turning point, $x = 1$,

$$7 = 1^2 + 4(1) + c$$

$$c = 2$$

The equation of the curve is

$$y = x^2 + 4x + 2.$$

$$(b) y = x^2 + 4x + 2$$

$$\frac{dy}{dx} = 2x + 4$$

$$\frac{d^2y}{dx^2} = 2$$

$$x^2 \frac{d^2y}{dx^2} + (x-1) \frac{dy}{dx} + y + 3 = 0$$

$$x^2(2) + (x-1)(2x+4) + x^2 + 4x + 2 + 3 = 0$$

$$2x^2 + 2x^2 + 2x - 4 + x^2 + 4x + 5 = 0$$

$$5x^2 + 6x + 1 = 0$$

$$(5x+1)(x+1) = 0$$

$$x = -\frac{1}{5} \text{ or } -1$$

$$12 (a) \frac{dy}{dx} = 3x^2 - 4x$$

$$y = \int (3x^2 - 4x) dx$$

$$y = x^3 - 2x^2 + c$$

The curve passes through the point

(-1, -2), thus $x = -1$ and $y = -2$.

$$-2 = (-1)^3 - 2(-1)^2 + c$$

$$-2 = -1 - 2 + c$$

$$c = 1$$

Hence, the equation of the curve is

$$y = x^3 - 2x^2 + 1.$$

$$(b) y = x^3 - 2x^2 + 1$$

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{d^2y}{dx^2} = 6x - 4$$

At the turning points,

$$\frac{dy}{dx} = 0$$

$$3x^2 - 4x = 0$$

$$x(3x - 4) = 0$$

$$x=0 \text{ or } x=\frac{4}{3}$$

At the turning point, $x=0$,

$$y=0^3-2(0)^2+1=1$$

Hence, $(0, 1)$ is a turning point.

$$\frac{d^2y}{dx^2}=6(0)-4=-2 \text{ (Negative)}$$

Hence, $(0, 1)$ is a maximum point.

At the turning point, $x=\frac{4}{3}$.

$$y=\left(\frac{4}{3}\right)^3-2\left(\frac{4}{3}\right)^2+1=-\frac{5}{27}$$

Hence, $\left(\frac{4}{3}, -\frac{5}{27}\right)$ is a turning point.

$$\frac{d^2y}{dx^2}=6\left(\frac{4}{3}\right)-4=4 \text{ (Positive)}$$

Hence, $\left(\frac{4}{3}, -\frac{5}{27}\right)$ is a minimum point.

$$\begin{aligned} \mathbf{13} \text{ (a)} \int_7^2 h(x) dx &= -\int_2^7 h(x) dx \\ &= -4 \end{aligned}$$

$$\text{(b)} \int_2^7 2h(x) dx - \int_2^7 p dx = 6$$

$$2(4) - p[x]_2^7 = 6$$

$$8 - p(7-2) = 6$$

$$8 - 5p = 6$$

$$-5p = -2$$

$$p = \frac{2}{5}$$

$$\mathbf{14} \text{ (a)} y = 6x - x^2$$

At the turning point where $x=2$,

$$y = 6(2) - 2^2 = 8$$

$$\frac{dy}{dx} = 6 - 2x = 6 - 2(2) = 2$$

Hence, the equation of the tangent is

$$y - 8 = 2(x - 2)$$

$$y - 8 = 2x - 4$$

$$y = 2x + 4$$

(b) At the turning point, $x=0$, $y=4$

At the turning point, $x=6$,

$$y = 2(6) + 4 = 16$$

Area of the shaded region

= (Area of the trapezium from $x=0$ to $x=6$) - (Area under the curve from $x=0$ to $x=6$)

$$= \frac{1}{2}(4+16)(6) - \int_0^6 (6x - x^2) dx$$

$$= 60 - \left[3x^2 - \frac{x^3}{3} \right]_0^6$$

$$= 60 - \left[3(6)^2 - \frac{6^3}{3} \right]$$

$$= 60 - (108 - 72)$$

$$= 60 - 36$$

$$= 24 \text{ units}^2$$

15 (a) At the x -axis, $y=0$

$$y = 27 - (x-2)^3$$

$$0 = 27 - (x-2)^3$$

$$(x-2)^3 = 27$$

$$x-2=3$$

$$x=5$$

Thus, P is point $(5, 0)$.

(b) Area of the shaded region

$$= \int_0^5 [27 - (x-2)^3] dx$$

$$= \left[27x - \frac{(x-2)^4}{4(1)} \right]_0^5$$

$$= \left[27(5) - \frac{3^4}{4} - \left(27(0) - \frac{(-2)^4}{4} \right) \right]$$

$$= \frac{459}{4} - \left(-\frac{16}{4} \right)$$

$$= \frac{475}{4} \text{ units}^2$$

$$\mathbf{16} \quad y = 2x \quad \dots (1)$$

$$y = 6x - x^2 \quad \dots (2)$$

Substitute (1) into (2) :

$$2x = 6x - x^2$$

$$x^2 - 6x + 2x = 0$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0 \text{ or } 4$$

At the turning point, $x=0$, $y=0$

At the turning point, $x=4$, $y=2(4)=8$

Thus, the points of intersection between the straight line $y = 2x$ and the curve

$$y = 6x - x^2 \text{ are } (0, 0) \text{ and } (4, 8).$$

Area of the shaded region B

$$= (\text{Area of triangle from } x = 0 \text{ to } x = 4) + (\text{Area under of curve from } x = 4 \text{ to } x = 6)$$

$$= \left(\frac{1}{2} \times 4 \times 8 \right) + \int_4^6 (6x - x^2) dx$$

$$= 16 + \left[3x^2 - \frac{x^3}{3} \right]_4^6$$

$$= 16 + \left[3(6)^2 - \frac{6^3}{3} - \left(3(4)^2 - \frac{4^3}{3} \right) \right]$$

$$= 16 + \left[36 - \frac{80}{3} \right]$$

$$= 16 + \frac{28}{3}$$

$$= \frac{76}{3} \text{ units}^2$$

Area under of curve from $x = 0$ to $x = 6$

$$= \int_0^6 (6x - x^2) dx$$

$$= \left[3x^2 - \frac{x^3}{3} \right]_0^6$$

$$= 3(6)^2 - \frac{6^3}{3} - 0$$

$$= 36 \text{ units}^2$$

Area of the shaded region A

$$= 36 - \frac{76}{3}$$

$$= \frac{32}{3} \text{ units}^2$$

Area of shaded region A : Area of shaded region B

$$= \frac{\text{Area of region } A}{\text{Area of region } B}$$

$$= \frac{32}{76}$$

$$= \frac{3}{19}$$

$$= \frac{8}{19}$$

$$= 8 : 19 \text{ [Shown]}$$

17 (a) Area of region A

$$= \int_1^2 y dx$$

$$= \int_1^2 \frac{8}{x^2} dx$$

$$= \int_1^2 8x^{-2} dx$$

$$= \left[\frac{8x^{-1}}{-1} \right]_1^2$$

$$= \left[-\frac{8}{x} \right]_1^2$$

$$= -\frac{8}{2} - \left(-\frac{8}{1} \right)$$

$$= -4 + 8$$

$$= 4 \text{ units}^2$$

(b) Area of region $B =$ Area of region C

$$\int_2^p 8x^{-2} dx = \int_p^5 8x^{-2} dx$$

$$\left[-\frac{8}{x} \right]_2^p = \left[-\frac{8}{x} \right]_p^5$$

$$-\frac{8}{p} - \left(-\frac{8}{2} \right) = -\frac{8}{5} - \left(-\frac{8}{p} \right)$$

$$-\frac{8}{p} + 4 = -\frac{8}{5} + \frac{8}{p}$$

$$\frac{16}{p} = \frac{28}{5}$$

$$p = \frac{5}{28} \times 16$$

$$p = 2\frac{6}{7}$$

18 (a) $y = (x-1)^2 = x^2 - 2x + 1$

$$\frac{dy}{dx} = 2x - 2$$

At the point $P(3, 4)$,

$$m = \frac{dy}{dx} = 2(3) - 2 = 4$$

Equation of the tangent is

$$y - 4 = 4(x - 3)$$

$$y - 4 = 4x - 12$$

$$y = 4x - 8$$

At point Q (x -axis),

$$y = 4x - 8$$

$$0 = 4x - 8$$

$$4x = 8$$

$$x = 2$$

Thus, Q is point $(2, 0)$.

(b) Area of the shaded region
 = (Area under the curve from $x = 1$ to $x = 3$) - (Area of triangle from $x = 2$ to $x = 3$)

$$= \int_1^3 (x-1)^2 dx - \left(\frac{1}{2} \times 1 \times 4\right)$$

$$= \left[\frac{(x-1)^3}{3(1)}\right]_1^3 - 2$$

$$= \frac{2^3}{3} - 0 - 2$$

$$= \frac{2}{3} \text{ unit}^2$$

19 (a) $y = x^3 - 3x + c$

$$\frac{dy}{dx} = 3x^2 - 3$$

At the minimum point P ,

$$\frac{dy}{dx} = 0$$

$$3x^2 - 3 = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = 1$$

Thus, P is point $(1, 0)$.

Substitute $x = 1$ and $y = 0$ into

$$y = x^3 - 3x + c$$

$$0 = 1^3 - 3(1) + c$$

$$c = 2$$

(b) Area of the shaded region

$$= \int_0^1 (x^3 - 3x + 2) dx$$

$$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1$$

$$= \frac{1}{4} - \frac{3}{2} + 2 - 0$$

$$= \frac{3}{4} \text{ unit}^2$$

20 $y = -x + 5 \quad \dots (1)$

$$y = x^2 - 2x + 5 \quad \dots (2)$$

Substitute (2) into (1):

$$x^2 - 2x + 5 = -x + 5$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \text{ or } x = 1$$

The straight line $y = -x + 5$ intersects the x -axis at $(5, 0)$.

At the turning point, $x = 0$, $y = -0 + 5 = 5$

At the turning point, $x = 1$,

$$y = -1 + 5 = 4$$

Thus, the points of intersection between the curve and the straight line are $(0, 5)$ and $(1, 4)$.

Area of the shaded region

= (Area under the curve from $x = 0$ to $x = 1$) + (Area of triangle from $x = 1$ to $x = 5$)

$$= \int_0^1 (x^2 - 2x + 5) dx + \frac{1}{2} \times (5-1) \times 4$$

$$= \left[\frac{x^3}{3} - x^2 + 5x\right]_0^1 + 8$$

$$= \left(\frac{1}{3} - 1 + 5\right) + 8$$

$$= \frac{37}{3} \text{ units}^2$$

21 $x = 3 \quad \dots (1)$

$$y^2 = 4(x+1) \quad \dots (2)$$

Substitute (1) into (2):

$$y^2 = 4(3+1) = 16$$

$$y = 4$$

Thus, B is point $(3, 4)$ and A is point $(0, 4)$.

$$y^2 = 4(x+1)$$

At the y -axis, $x = 0$.

$$y^2 = 4(0+1)$$

$$y = 2$$

Thus, the curve intersects the y -axis at $(0, 2)$.

Area of the shaded region

$$= \int_2^4 x dy$$

$$= \int_2^4 \left(\frac{y^2}{4} - 1\right) dy$$

$$= \left[\frac{y^3}{12} - y\right]_2^4$$

$$= \frac{4^3}{12} - 4 - \left(\frac{2^3}{12} - 2\right)$$

$$y^2 = 4(x+1)$$

$$\frac{y^2}{4} = x+1$$

$$x = \frac{y^2}{4} - 1$$

$$= \frac{4}{3} - \left(-\frac{4}{3}\right)$$

$$= \frac{8}{3} \text{ units}^2$$

22 (a) Point $(k, 3)$ lies on the straight line $y = 5 - x$.

$$\text{Thus, } 3 = 5 - k \Rightarrow k = 2.$$

(b) The straight line $y = 5 - x$ intersects the y -axis at $(0, 5)$.

The curve $y = (1 - x)(x - 5)$ intersects the x -axis $(1, 0)$ and $(5, 0)$.

Area of the shaded region
 = (Area of trapezium from $x = 0$ to $x = 2$) - Area under the curve from $x = 1$ to $x = 2$)

$$= \frac{1}{2}(5+3)(2) - \int_1^2 y \, dx$$

$$= 8 - \int_1^2 (1-x)(x-5) \, dx$$

$$= 8 - \int_1^2 (-x^2 + 6x - 5) \, dx$$

$$= 8 - \left[-\frac{x^3}{3} + 3x^2 - 5x \right]_1^2$$

$$= 8 - \left[-\frac{2^3}{3} + 3(2)^2 - 5(2) - \left(-\frac{1}{3} + 3 - 5 \right) \right]$$

$$= 8 - \left[-\frac{2}{3} - \left(-\frac{7}{3} \right) \right]$$

$$= \frac{19}{3} \text{ units}^2$$

23 Area of the shaded region

$$= \int_{-2}^2 [(-x^2 + 2x + 7) - (x^2 + 2x - 1)] \, dx$$

$$= \int_{-2}^2 (-2x^2 + 8) \, dx$$

$$= \left[-\frac{2x^3}{3} + 8x \right]_{-2}^2$$

$$= \left[-\frac{2(2)^3}{3} + 8(2) - \left(-\frac{2}{3}(-2)^3 + 8(-2) \right) \right]$$

$$= \frac{32}{3} - \left(-\frac{32}{3} \right)$$

$$= 21\frac{1}{3} \text{ units}^2$$

24 $y = x(x - 2) = x^2 - 2x$

$$y^2 = (x^2 - 2x)^2 = (x^4 - 4x^3 + 4x^2)$$

Generated volume

$$= \pi \int_0^3 y^2 \, dx$$

$$= \pi \int_0^3 (x^4 - 4x^3 + 4x^2) \, dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3} \right]_0^3$$

$$= \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_0^3$$

$$= \pi \left(\frac{3^5}{5} - 3^4 + 36 - 0 \right)$$

$$= \pi \left(\frac{243}{5} - 81 + 36 \right)$$

$$= \frac{18}{5} \pi \text{ units}^3$$

25 Generated volume

$$= \pi \int_{-2}^1 x^2 \, dy$$

$$= \pi \int_{-2}^1 (y^2 + 4)^2 \, dy$$

$$= \pi \int_{-2}^1 (y^4 + 8y^2 + 16) \, dy$$

$$= \pi \left[\frac{y^5}{5} + \frac{8y^3}{3} + 16y \right]_{-2}^1$$

$$= \pi \left[\frac{1}{5} + \frac{8}{3} + 16 - \left(-\frac{32}{5} + \frac{8}{3}(-8) - 32 \right) \right]$$

$$= \pi \left[\frac{283}{15} - \left(-\frac{896}{15} \right) \right]$$

$$= \frac{393}{5} \pi \text{ units}^3$$

26 Generated volume

$$= \pi \int_2^5 x^2 \, dy$$

$$= \pi \int_2^5 (25 - y^2) \, dy$$

$$= \pi \left[25y - \frac{y^3}{3} \right]_2^5$$

$$\begin{aligned}
&= \pi \left[25(5) - \frac{5^3}{3} - \left(25(2) - \frac{2^3}{3} \right) \right] \\
&= \pi \left[125 - \frac{125}{3} - \left(50 - \frac{8}{3} \right) \right] \\
&= \pi \left(\frac{250}{3} - \frac{142}{3} \right) \\
&= 36\pi \text{ units}^3
\end{aligned}$$

27 (a) Generated volume

$$\begin{aligned}
&= \pi \int_0^4 x^2 dy \\
&= \pi \int_0^4 (4-y) dy \\
&= \pi \left[4y - \frac{y^2}{2} \right]_0^4 \\
&= \pi(16 - 8 - 0) \\
&= 8\pi \text{ units}^3
\end{aligned}$$

(b) The curve intersects the x -axis at the points $(-2, 0)$ and $(2, 0)$.

Generated volume

$$\begin{aligned}
&= 2\pi \int_0^2 y^2 dx \\
&= 2\pi \int_0^2 (4-x^2)^2 dx \\
&= 2\pi \int_0^2 (16-8x^2+x^4) dx \\
&= 2\pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2 \\
&= 2\pi \left(16(2) - \frac{8(2)^3}{3} + \frac{2^5}{5} - 0 \right) \\
&= 2\pi \left(32 - \frac{64}{3} + \frac{32}{5} - 0 \right) \\
&= \frac{512}{15}\pi \text{ units}^3
\end{aligned}$$

28 (a) The curve intersects the x -axis at $(2, 0)$ and the y -axis at the point $(0, 5)$.

Generated volume

$$\begin{aligned}
&= \pi \int_0^4 y^2 dx \\
&= \pi \int_0^4 \frac{16-x^2}{4} dx \\
&= \frac{1}{4}\pi \int_0^4 (16-x^2) dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\pi \left[16x - \frac{x^3}{3} \right]_0^4 \\
&= \frac{1}{4}\pi \left[16(4) - \frac{4^3}{3} - 0 \right] \\
&= \frac{1}{4}\pi \left(\frac{128}{3} \right) \\
&= \frac{32}{3}\pi \text{ units}^2
\end{aligned}$$

(b) Generated volume

$$\begin{aligned}
&= \pi \int_0^4 x^2 dy \\
&= \pi \int_0^2 (16-4y^2) dy \\
&= \pi \left[16y - \frac{4y^3}{3} \right]_0^2 \\
&= \pi \left(32 - \frac{32}{3} \right) \\
&= \frac{64}{3}\pi \text{ units}^3
\end{aligned}$$

29 $y = 3\sqrt{x-1}$

$$y^2 = 9(x-1)$$

At the x -axis, $y = 0$

$$0^2 = 9(x-1)$$

$$x = 1$$

Generated volume = $\frac{9}{2}\pi \text{ units}^3$

$$\pi \int_1^k y^2 dx = \frac{9}{2}\pi$$

$$\int_1^k 9(x-1) dx = \frac{9}{2}$$

$$\int_1^k (x-1) dx = \frac{1}{2}$$

$$\left[\frac{x^2}{2} - x \right]_1^k = \frac{1}{2}$$

$$\frac{k^2}{2} - k - \left(\frac{1}{2} - 1 \right) = \frac{1}{2}$$

$$\frac{k^2}{2} - k + \frac{1}{2} = \frac{1}{2}$$

$$\frac{k^2}{2} - k = 0$$

$$k^2 - 2k = 0$$

$$k(k-2) = 0$$

$$k = 2$$

30 Generated volume = 2π

$$\pi \int_{-3}^k x^2 dy = 2\pi$$

$$\int_{-3}^k (y+3) dy = 2$$

$$\left[\frac{y^2}{2} + 3y \right]_{-3}^k = 2$$

$$\frac{k^2}{2} + 3k - \left(\frac{9}{2} - 9 \right) = 2$$

$$k^2 + 6k - (9 - 18) = 4$$

$$k^2 + 6k + 5 = 0$$

$$(k+1)(k+5) = 0$$

$$k = -5 \text{ or } k = -1$$

$k = -5$ is not accepted.

$\therefore k = -1$

31 (a) $3y = x \dots (1)$

$$x^2 + 7y^2 = 16 \dots (2)$$

Substitute (1) into (2) :

$$(3y)^2 + 7y^2 = 16$$

$$16y^2 = 16$$

$$y^2 = 1$$

$$y = \pm 1$$

$y = -1$ is not accepted.

$\therefore y = 1$

From (1) :

$$3(1) = x \Rightarrow x = 3$$

Thus, A is point (3, 1).

$$x^2 + 7y^2 = 16$$

At the x -axis, $y = 0$.

$$x^2 + 7(0)^2 = 16$$

$$x = 4$$

Thus, B is point (4, 0).

(b) $x^2 + 7y^2 = 16$

$$y^2 = \frac{16-x^2}{7}$$

Generated volume

= (Volume of cone generated by the straight line from $x = 0$ to $x = 3$) +
Volume generated by the curve from $x = 3$ to $x = 4$)

$$= \frac{1}{3}\pi r^2 h + \pi \int_3^4 y^2 dx$$

$$= \frac{1}{3}\pi(1)(3) + \pi \int_3^4 \left(\frac{16-x^2}{7} \right) dx$$

$$= \pi + \frac{1}{7}\pi \left[16x - \frac{x^3}{3} \right]_3^4$$

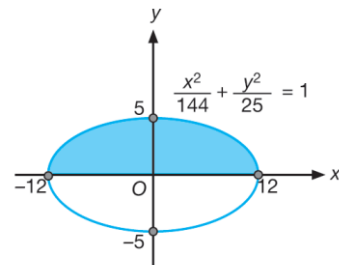
$$= \pi + \frac{1}{7}\pi \left[16(4) - \frac{64}{3} - \left(48 - \frac{27}{3} \right) \right]$$

$$= \pi + \frac{1}{7}\pi \left[16(4) - \frac{64}{3} - \left(48 - \frac{27}{3} \right) \right]$$

$$= \pi + \frac{1}{7}\left(\frac{11}{3} \right)\pi$$

$$= \frac{32}{21}\pi \text{ units}^3$$

32



Smart Strategy

The volume of the rugby ball is given by the generated volume when the shaded region is rotated through 360° about the x -axis.

HOT TIPS

$\frac{x^2}{144} + \frac{y^2}{25} = 1$ is the equation of an ellipse.

$$\frac{x^2}{144} + \frac{y^2}{25} = 1$$

$$\frac{y^2}{25} = 1 - \frac{x^2}{144}$$

$$y^2 = 25 \left(1 - \frac{x^2}{144} \right)$$

Volume of the rugby ball
= Generated volume

$$= \pi \int_{-12}^{12} y^2 dx$$

$$= \pi \int_{-12}^{12} 25 \left(1 - \frac{x^2}{144} \right) dx$$

$$= 25\pi \left[x - \frac{x^3}{432} \right]_{-12}^{12}$$

$$\begin{aligned}
&= 25\pi \left[12 - \frac{12^3}{432} - \left[-12 - \frac{(-12)^3}{432} \right] \right] \\
&= 25\pi [8 - (-8)] \\
&= 400\pi \text{ units}^3 \\
&= 400 \times 3.142 \\
&= 1\,256.8 \text{ units}^3
\end{aligned}$$

Hence, the volume of the rugby ball is $1\,256.8 \text{ cm}^3$.

SPM Spot

1 $\frac{dy}{dx} = (2x+1)^3$

$$y = \int (2x+1)^3 dx$$

$$y = \frac{(2x+1)^4}{4(2)} + c$$

$$y = \frac{(2x+1)^4}{8} + c$$

The curve passes through the point

$$\left(\frac{1}{2}, -3\right).$$

$$-3 = \frac{\left[2\left(\frac{1}{2}\right) + 1\right]^4}{8} + c$$

$$-3 = 2 + c$$

$$c = -5$$

Hence, the equation of the curve is

$$y = \frac{(2x+1)^4}{8} - 5.$$

2 (a) (i) $y = x^2 - 6x + 16 \dots (1)$

$$y = 6x - x^2 \dots (2)$$

Substitute (1) into (2) :

$$x^2 - 6x + 16 = 6x - x^2$$

$$2x^2 - 12x + 16 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0$$

$$x = 2 \text{ or } x = 4$$

From (2):

$$\text{When } x = 2, y = 6(2) - 2^2 = 8$$

$$\text{When } x = 4, y = 6(4) - 4^2 = 8$$

Hence, A is point (2, 8) and B is point (4, 8).

(ii) Area bounded by the two curves

$$= \int_2^4 (6x - x^2) dx - \int_2^4 (x^2 - 6x + 16) dx$$

$$= \int_2^4 (6x - x^2 - x^2 + 6x - 16) dx$$

$$= \int_2^4 (12x - 2x^2 - 16) dx$$

$$= \left[6x^2 - \frac{2x^3}{3} - 16x \right]_2^4$$

$$= 6(4)^2 - \frac{2}{3}(4)^3 - 16(4)$$

$$- \left[6(2)^2 - \frac{2}{3}(2)^3 - 16(2) \right]$$

$$= -10\frac{2}{3} - \left(-13\frac{1}{3}\right)$$

$$= 2\frac{2}{3} \text{ units}^2$$

(b) Generated volume = $42\frac{1}{2}\pi$

$$\pi \int_3^k y^2 dx - \text{Volume of cone} = 42\frac{1}{2}\pi$$

$$\pi \int_3^k (x+6) dx - \frac{1}{3}\pi r^2 h = \frac{85}{2}\pi$$

$$\left[\frac{x^2}{2} + 6x \right]_3^k - \frac{1}{3}(3)^2(k-3) = \frac{85}{2}$$

$$\frac{k^2}{2} + 6k - \left(\frac{9}{2} + 18\right) - 3k + 9 = \frac{85}{2}$$

$$\frac{k^2}{2} + 3k - 56 = 0$$

$$k^2 + 6k - 112 = 0$$

$$(k-8)(k+14) = 0$$

$$k = 8 \text{ or } k = -14$$

$$k = -14 \text{ is not accepted.}$$

$$\therefore k = 8$$

3 (a) $y^2 = -x + 9$

$$y = \sqrt{-x+9}$$

$$y = (-x+9)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(-x+9)^{-\frac{1}{2}}(-1)$$

$$= -\frac{1}{2\sqrt{-x+9}}$$

$$\text{When } x = 5, \frac{dy}{dx} = -\frac{1}{2\sqrt{-5+9}} = -\frac{1}{4}$$

Thus, the gradient of the tangent =

$$-\frac{1}{4}$$

Equation of the tangent is

$$y - 2 = -\frac{1}{4}(x - 5)$$

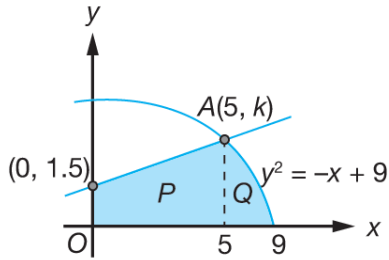
$$4y - 8 = -x + 5$$

$$4y = -x + 13$$

$$y = \sqrt{-5 + 9} = 2$$

Thus, $k = 2$

(b)



Area of the shaded region

= Area P + Area Q

$$= \frac{1}{2}(2 + 1.5)(5) + \int_5^9 (-x + 9)^{\frac{1}{2}} dx +$$

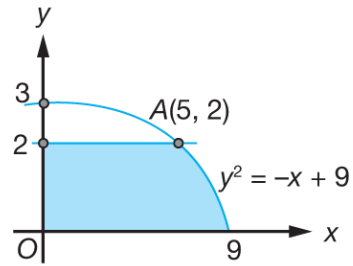
$$= 8\frac{3}{4} + \left[\frac{(-x + 9)^{\frac{3}{2}}}{-\frac{3}{2}} \right]_5^9$$

$$= 8\frac{3}{4} - \frac{2}{3} \left[(-9 + 9)^{\frac{3}{2}} - (-5 + 9)^{\frac{3}{2}} \right]$$

$$= 8\frac{3}{4} - \frac{2}{3} [0 - 8]$$

$$= 14\frac{1}{12} \text{ units}^2$$

(c)



$$V_y = \pi \int_0^2 x^2 dy$$

$$= \pi \int_0^2 (-y^2 + 9)^2 dy$$

$$= \pi \int_0^2 (y^4 - 18y^2 + 81) dy$$

$$= \pi \left[\frac{y^5}{5} - 18 \times \frac{y^3}{3} + 81y \right]_0^2$$

$$= \pi \left[\frac{2^5}{5} - 6(2)^3 + 81(2) - 0 \right]$$

$$= 120\frac{2}{5} \pi \text{ units}^3$$